Asymptotic equidistribution for partition statistics and topological invariants

joint work with William Craig and Joshua Males

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Motivation

A *partition* $\lambda$ of a positive integer $n$ is a list of non-increasing positive integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$. 

Example

For $n = 4$ the possible partitions are given by $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. Thus we have $p(4) = 5$. 

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Motivation

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Equidistribution properties of certain objects are a central theme studied by many authors in many mathematical fields.
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Asymptotic equidistribution

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Suppose that $c(n)$ is an arithmetic counting function e.g. $c(n) = p(n)$. Suppose $s(\lambda)$ is an integer valued partition invariant
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Suppose that \( c(n) \) is an arithmetic counting function e.g. \( c(n) = p(n) \). Suppose \( s(\lambda) \) is an integer valued partition invariant and let

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c(a, b; n) := \#\{\text{partitions of } n : s(\lambda) \equiv a \pmod{b}\}.
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$$c(a, b; n) := \#\{\text{partitions of } n : s(\lambda) \equiv a \pmod{b}\}.$$  

To say that equidistribution holds is to say that

$$c(a, b; n) \sim \frac{1}{b}c(n)$$  

as $n \to \infty$. 
Examples for recently studied modular typed objects:
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1. Asymptotic equidistribution of partition ranks (Males).
2. Asymptotic equidistribution results for partitions into $k$-th powers (Ciolan).
3. Asymptotic equidistribution for Hodge numbers and Betti numbers of certain Hilbert schemes of surfaces (Gillman–Gonzalez–Ono–Rolen–Schoenbauer).
4. Asymptotic equidistribution of partitions whose parts are values of a given polynomial (Zhou).
Each partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) has a \textit{Ferrers–Young diagram:}
Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ has a *Ferrers–Young diagram*:

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \ldots & \bullet & \leftarrow & \lambda_1 \text{ many nodes} \\
\bullet & \bullet & \ldots & \bullet & \leftarrow & \lambda_2 \text{ many nodes} \\
\vdots & \vdots & \vdots & \vdots & \leftarrow & \vdots \\
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The node in row \( k \) and column \( j \) has hook length

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h(k,j) := (\lambda_k - k) + (\lambda'_j - j) + 1,
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Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ has a Ferrers–Young diagram:

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\end{align*}

The node in row $k$ and column $j$ has hook length $h(k, j)$:

$$h(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1,$$

where $\lambda'_j := \# \text{ nodes in column } j$. 
Let $\mathcal{H}_t(\lambda)$ denote the multiset of $t$-hooks, those hook lengths which are multiples of a fixed positive integer $t$, of a partition $\lambda$. 
Let $\mathcal{H}_t(\lambda)$ denote the multiset of $t$-hooks, those hook lengths which are multiples of a fixed positive integer $t$, of a partition $\lambda$. Let

$$p_t^e(n) := \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is even}\},$$

$$p_t^o(n) := \#\{\lambda \text{ a partition of } n : \#\mathcal{H}_t(\lambda) \text{ is odd}\}.$$
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**Craig–Pun:**
For even $t$ the partitions of $n$ are asymptotically equidistributed between these two subsets, for odd $t$ they are not.
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**Craig–Pun:**
For even $t$ the partitions of $n$ are asymptotically equidistributed between these two subsets, for odd $t$ they are not.

**Bringmann–Craig–Males–Ono:**
On arithmetic progressions modulo odd primes $t$-hooks are not asymptotically equidistributed. The Betti numbers of two specific Hilbert schemes are asymptotically equidistributed.
Wright’s Circle Method

Hardy–Ramanujan, 1918

\[ p(n) \sim \frac{1}{4\sqrt{3n}} \cdot e^{\pi \sqrt{\frac{2n}{3}}}, \quad \text{as } n \to \infty. \]
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The essence of Wright’s method is to use Cauchy’s theorem. We have

\[ \mathcal{A}(\tau) := \sum_{n \geq 0} a(n) q^n \quad \rightarrow \quad a(n) = \frac{1}{2\pi i} \int_C \frac{\mathcal{A}(q)}{q^{n+1}} dq, \]

where \( q = e^{2\pi i \tau} \).
Wright’s Circle Method

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\[ p(n) \sim \frac{1}{4\sqrt{3n}} \cdot e^{\pi \sqrt{\frac{2n}{3}}} + \epsilon(n), \quad \text{as } n \to \infty. \]

The essence of Wright’s method is to use Cauchy’s theorem. We have

\[ A(\tau) := \sum_{n \geq 0} a(n) q^n \quad \rightarrow \quad a(n) = \frac{1}{2\pi i} \int_C \frac{A(q)}{q^{n+1}} dq, \]

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One then splits the integral into two arcs, the major arc and minor arc.
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Following Wright and the work of Ngo–Rhoades, Bringmann–Craig–Males–Ono proved the following variant of Wright’s Circle Method.
Let $M > 0$ be a fixed constant and $z = x + iy \in \mathbb{C}$, with $x > 0$ and $|y| < \pi$. 
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Consider the following hypotheses:
Variant of Wright’s Circle Method

Let $M > 0$ be a fixed constant and $z = x + iy \in \mathbb{C}$, with $x > 0$ and $|y| < \pi$.

Consider the following hypotheses:

(i) As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have

$$F(e^{-z}) = z^B e^{A/z} (\alpha_0 + O_M(|z|)),$$

where $\alpha_0 \in \mathbb{C}$, $A \in \mathbb{R}^+$, and $B \in \mathbb{R}$.

(ii) As $z \to 0$ in the bounded cone $Mx \leq |y| < \pi$ (minor arc), we have

$$|F(e^{-z})| \ll M e^{1 \Re(z) (A - \kappa)}$$

for some $\kappa \in \mathbb{R}^+$. 
Let $M > 0$ be a fixed constant and $z = x + iy \in \mathbb{C}$, with $x > 0$ and $|y| < \pi$.
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(i) As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have

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(ii) As $z \to 0$ in the bounded cone $Mx \leq |y| < \pi$ (minor arc), we have

$$|F(e^{-z})| \ll_M e^{\frac{1}{\Re(z)}(A - \kappa)},$$

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near 1.

If (i) and (ii) hold, then as $n \to \infty$ we have

$$c(n) = n^{\frac{1}{4}}(-2B-3) e^{2\sqrt{A}n} \left( p_0 + O \left( n^{-\frac{1}{2}} \right) \right),$$
Suppose that $F(q)$ is analytic for $q = e^{-z}$ where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$, and suppose that $F(q)$ has an expansion $F(q) = \sum_{n=0}^{\infty} c(n)q^n$ near 1.

If (i) and (ii) hold, then as $n \to \infty$ we have

$$c(n) = n^{\frac{1}{4}}(-2B-3) e^{2\sqrt{An}} \left( p_0 + O \left( n^{-\frac{1}{2}} \right) \right),$$

where $p_0 = \alpha_0 \frac{\sqrt{A}^{B+\frac{1}{2}}}{2\sqrt{\pi}}$. 

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Asymptotic equidistribution

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Setting of Central Theorem

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Furthermore let $\zeta = \zeta^a_b := e^{\frac{2\pi ia}{b}}$ ($b \geq 2$ and $0 \leq a < b$).
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Furthermore let \( \zeta = \zeta^a_b := e^{\frac{2\pi i a}{b}} \) (\( b \geq 2 \) and \( 0 \leq a < b \)).

Assume that we have a generating function on arithmetic progressions \( a \) (mod \( b \)) given by

\[
H(a, b; q) := \sum_{n \geq 0} c(a, b; n)q^n,
\]

for some coefficients \( c(a, b; n) \)
Setting of Central Theorem

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H(a, b; q) = \frac{1}{b} \sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; q)
\]

for some generating functions \( H(\zeta; q) \), with

\[
H(q) := H(1; q) = \sum_{n \geq 0} c(n)q^n.
\]
Setting of Central Theorem

Let \( H(a, b; q) \) and \( H(\zeta; q) \) be analytic on \(|q| < 1\) such that the above holds.
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Let $H(a, b; q)$ and $H(\zeta; q)$ be analytic on $|q| < 1$ such that the above holds.

Let $C = C_n$ be a sequence of circles centered at the origin inside the unit disk with radii $r_n \to 1$ as $n \to \infty$ that loops around zero exactly once.
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For $0 \leq \theta < \frac{\pi}{2}$ let

$$D_\theta := \{ z = re^{i\alpha} : r \geq 0 \text{ and } |\alpha| \leq \theta \}.$$
Setting of Central Theorem

Let $H(a, b; q)$ and $H(\zeta; q)$ be analytic on $|q| < 1$ such that the above holds.

Let $C = C_n$ be a sequence of circles centered at the origin inside the unit disk with radii $r_n \to 1$ as $n \to \infty$ that loops around zero exactly once.

For $0 \leq \theta < \frac{\pi}{2}$ let

$$D_\theta := \{z = re^{i\alpha}: r \geq 0 \text{ and } |\alpha| \leq \theta\}.$$
Setting of Central Theorem

For $\theta > 0$, let $\tilde{C} := C \cap D_{\theta}$ and $C \setminus \tilde{C}$ be arcs such that the following hypotheses hold.

1. As $z \to 0$ outside of $D_{\theta}$, we have $b^{-1} \sum_{j=1}^{b-1} \zeta_j - a_j b H(\zeta_j b; e^{-z}) = O(H(1; e^{-z}))$.
2. As $z \to 0$ in $D_{\theta}$, we have for each $1 \leq j \leq b-1$ that $H(\zeta_j b; e^{-z}) = o(H(1; e^{-z}))$.
3. As $n \to \infty$, we have $c(n) \sim \frac{1}{2} \pi i \int_{\tilde{C}} H(1; q) q^{n+1} dq$. 

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Setting of Central Theorem

For $\theta > 0$, let $\tilde{C} := C \cap D_\theta$ and $C \setminus \tilde{C}$ be arcs such that the following hypotheses hold.

(1) As $z \to 0$ outside of $D_\theta$, we have

$$\sum_{j=1}^{b-1} \zeta^j b^{-aj} H(\zeta^j b; e^{-z}) = O \left( H(1; e^{-z}) \right).$$

(2) As $z \to 0$ in $D_\theta$, we have for each $1 \leq j \leq b-1$ that

$$H(\zeta^j b; e^{-z}) = o \left( H(1; e^{-z}) \right).$$

(3) As $n \to \infty$, we have

$$c(n) \sim \frac{1}{2} \pi i \int_{\tilde{C}} H(1; q) q^n + 1 dq.$$
Setting of Central Theorem

For \( \theta > 0 \), let \( \tilde{C} := C \cap D_\theta \) and \( C \setminus \tilde{C} \) be arcs such that the following hypotheses hold.

1. As \( z \to 0 \) outside of \( D_\theta \), we have
   \[
   \sum_{j=1}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; e^{-z}) = O \left( H(1; e^{-z}) \right).
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2. As \( z \to 0 \) in \( D_\theta \), we have for each \( 1 \leq j \leq b - 1 \) that
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   \[ H(\zeta_b^j; e^{-z}) = o \left( H(1; e^{-z}) \right). \]

3. As $n \to \infty$, we have
   \[ c(n) \sim \frac{1}{2\pi i} \int_{\tilde{C}} \frac{H(1; q)}{q^{n+1}} dq. \]
Central Theorem

As $n \to \infty$, we have

$$c(a, b; n) \sim \frac{1}{b} c(n).$$
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$$c(a, b; n) \sim \frac{1}{b} c(n).$$

In particular, if $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO we have that

$$c(a, b; n) \sim \frac{1}{b} c(n) \sim \frac{1}{b} n^{\frac{1}{4}(2B-3)} e^{2\sqrt{An}} \left( p_0 + O \left( n^{-\frac{1}{2}} \right) \right)$$

as $n \to \infty$. 

Idea of the proof

1. Use Cauchy's theorem and the decomposition of $H(a, b; q)$ to obtain

$$c(a, b; n) = \frac{1}{b} \left[ \frac{1}{2\pi i} \int_C \frac{1}{q^{n+1}} \sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; q) dq \right].$$
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2. Break down the integral over $C$ into the components $\tilde{C}$ and $C \setminus \tilde{C}$ and look at each of them separately.
Idea of the proof

1. Use Cauchy’s theorem and the decomposition of $H(a, b; q)$ to obtain

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2. Break down the integral over $C$ into the components $\tilde{C}$ and $C \setminus \tilde{C}$ and look at each of them separately.

3. Along $C \setminus \tilde{C}$ we have by conditions (1) and (3) that as $n \to \infty$

$$\frac{1}{2\pi i} \int_{C \setminus \tilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta_b^j; q)}{q^{n+1}} dq = o \left( \frac{1}{2\pi i} \int_{\tilde{C}} \frac{H(1; q)}{q^{n+1}} dq \right).$$
On \( \tilde{C} \) we obtain with (2) and as \( n \to \infty \) that

\[
\frac{1}{2\pi i} \int_{\tilde{C}} \sum_{j=0}^{b-1} \zeta_b^{-aj} H(\zeta^j_b; q) \frac{dq}{q^{n+1}} \sim \frac{1}{2\pi i} \int_{\tilde{C}} \frac{H(1; q)}{q^{n+1}} dq.
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Idea of the proof

4 On \( \tilde{C} \) we obtain with (2) and as \( n \to \infty \) that

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\]

5 The first claim follows by combining the estimates along \( \tilde{C} \) and \( C \setminus \tilde{C} \).
Idea of the proof

4 On $\tilde{C}$ we obtain with (2) and as $n \to \infty$ that

$$\frac{1}{2\pi i} \int_{\tilde{C}} \frac{\sum_{j=0}^{b-1} \zeta_{b}^{-aj} H(\zeta_{b}^{j}; q)}{q^{n+1}} dq \sim \frac{1}{2\pi i} \int_{\tilde{C}} \frac{H(1; q)}{q^{n+1}} dq.$$ 

5 The first claim follows by combining the estimates along $\tilde{C}$ and $C \setminus \tilde{C}$.

6 If we assume $H(1; q)$ and $H(\zeta_{b}^{j}; q)$ satisfy the hypotheses of BCMO, then (1) – (3) are satisfied and the result follows by the asymptotic for $c(n)$ in BCMO.
Let \( 0 \leq a < b \) and \( b \geq 2 \). Assume that \( H(1; q) \) and \( H(\zeta; q) \) satisfy the conditions of BCMO. Then for sufficiently large \( n_1, n_2 \) we have

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Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO. For large enough $n$, we have

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3. spt-function (Dawsey–Masri)
Ramanujan congruences, 1921

For \( n \geq 0 \) we have

\[
\begin{align*}
  p(5n + 4) & \equiv 0 \pmod{5}, \\
  p(7n + 5) & \equiv 0 \pmod{7}, \\
  p(11n + 6) & \equiv 0 \pmod{11}.
\end{align*}
\]

Example

The rank of a partition \( \lambda \) is the largest part minus the number of parts.

The ranks of the partitions of 4:

<table>
<thead>
<tr>
<th>partition</th>
<th>rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4)</td>
<td>3</td>
</tr>
<tr>
<td>(3,1)</td>
<td>1</td>
</tr>
<tr>
<td>(2,2)</td>
<td>0</td>
</tr>
<tr>
<td>(2,1,1)</td>
<td>-1</td>
</tr>
<tr>
<td>(1,1,1,1)</td>
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C.—Craig—Males, 2021

Let 0 \leq a < b and b \geq 2. Then as n \to \infty we have that

\[ N(a, b; n) = \frac{1}{b} p(n) \left( 1 + O\left( n^{-\frac{1}{2}} \right) \right). \]
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The equidistribution of \( N(a, b; n) \) was already proven by Males in 2021 using Ingham’s Tauberian theorem.
The crank

\[
\text{crank}(\lambda) := \begin{cases} 
\text{largest part of } \lambda & \text{if } \omega(\lambda) = 0, \\
\mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0
\end{cases}
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The first residual crank

An overpartition is a partition where the first occurrence of each distinct number may be overlined.
The first residual crank

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Example
The overpartitions of 4 are given by

(4), (4), (3, 1), (3, 1), (3, 1), (3, 1), (2, 2), (2, 2), (2, 1, 1), (2, 1, 1), (2, 1, 1), (2, 1, 1), (1, 1, 1, 1), (1, 1, 1, 1).
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**Example**

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The first residual crank of an overpartition is given by the crank of the subpartition consisting of the non-overlined parts.
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So the first residual crank of $(2, \bar{1}, 1)$ is given by the crank of $(2, 1)$ which equals 0.
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Let \(0 \leq a < b\) and \(b \geq 2\). Then as \(n \to \infty\) we have that

\[
\bar{M}(a, b; n) = \frac{1}{8bn} e^{\frac{\pi}{\sqrt{n}}} \left(1 + O \left(n^{-\frac{1}{2}}\right)\right).
\]
A plane partition of $n$ is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j, k \geq 1$, that is non-increasing in both variables, i.e., 
$\pi_{j,k} \geq \pi_{j+1,k}$, $\pi_{j,k} \geq \pi_{j,k+1}$ for all $j$ and $k$, and fulfils $|\Lambda| := \sum_{j,k} \pi_{j,k} = n$. 

Example 
For $n = 3$ we have the plane partitions:

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1
2 1 2
1 3

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A plane partition may be represented visually by the placement of a stack of $\pi_j, k$ unit cubes above the point $(j, k)$ in the plane, giving a three-dimensional solid. The sum $|\Lambda|$ then describes the number of cubes of which the plane partition consists.
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We have that $\text{pp}(0, 2; 3) = 2$ and $\text{pp}(1, 2; 3) = 4$. 
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C.—Craig—Males, 2021

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**C. Craig Males, 2021**

Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$\text{pp}(a, b; n) \sim \frac{1}{b} \text{pp}(n) \sim \frac{1}{b} \frac{\zeta(3)^{\frac{7}{56}}}{\sqrt{12\pi}} \left( \frac{n}{2} \right)^{-\frac{25}{36}} \exp \left( 3\zeta(3)^{\frac{1}{3}} \left( \frac{n}{2} \right)^{\frac{2}{3}} + \zeta'(-1) \right).$$
Betti numbers count the dimension of certain vector spaces of differential forms of a manifold.
Betti numbers of Hilbert schemes

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For a Hilbert scheme $X$, let $b_j(X) := \dim(H_j(X, \mathbb{Q}))$ be the Betti numbers, where $H_j(X, \mathbb{Q})$ denotes the $j$-th homology group of $X$ with rational coefficients.
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We define the Hilbert schemes

\[
\begin{align*}
X_1 &:= \text{Hilb}^{n,n+1,n+2}(0), \\
X_2 &:= \text{Hilb}^{n,n+2}(0), \\
X_3 &:= \text{Hilb}^{n,n+2}(\mathbb{C}^2)_\text{tr}, \\
X_4 &:= \hat{M}^m(c_N),
\end{align*}
\]

where $m \in \mathbb{N}$ and $c_N$ is some prescribed homological data.
Let $0 \leq a < b$ with $b \geq 2$ and

$$d(a, b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd}, \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even}, \\ 0 & \text{if } a \text{ is odd and } b \text{ is even}. \end{cases}$$
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Then as $n \to \infty$ we have that

$$\frac{1}{2} B(a, b; X_1) \sim B(a, b; X_2) \sim B(a, b; X_3) = \frac{d(a, b)\sqrt{3}}{4\pi^2} e^{\pi \sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right)$$
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and
\[
B(a, b; X_4) = \frac{d(a, b) n^{\frac{m-2}{2}}}{6^{\frac{1-m}{2}} 2\sqrt{2} c_m \pi^m} e^{\pi \sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),
\]
where $\prod_{j=1}^{m} \frac{1}{1 - e^{-jz}} = \frac{1}{c_m z^m} + O(z^{-m+1})$. 
A particular scheme of Göttsche

Let $K$ be an algebraically closed field.
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Let $K$ be an algebraically closed field. Let $m$ be the maximal ideal in $K[[x, y]]$, and define

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$\text{−} \text{Craig} \text{−} \text{Males, 2021}$

Let $0 \leq a < b$ and $b \geq 2$. As $n \to \infty$ we have that $v(a, b; n) = \frac{1}{b} p(n) \left(1 + O \left(\frac{1}{n^{\frac{1}{2}}}\right)\right)$.
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C.—Craig—Males, 2021

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Proof for crank

Using orthogonality of roots of unity we have

$$\sum_{n \geq 0} M(a, b; n) q^n = \frac{1}{b} \sum_{n \geq 0} p(n) q^n + \frac{1}{b} \sum_{j=1}^{b-1} \zeta_{b}^{-aj} C \left( \zeta_{b}^j; q \right),$$

where

$$C \left( \zeta; q \right) := \left( \frac{q; q}{\eta(q)} \right) \frac{\eta(q)}{\eta(q^{\mu})} F_1 \left( \zeta q; q \right) F_1 \left( \zeta^{-1}; q \right),$$

with

$$\left( \frac{q}{q} \right) \eta(q) := \prod_{\ell=1}^{\infty} (1 - q^\ell)$$

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with \((q; q)_{\infty} := \prod_{\ell=1}^{\infty} (1 - q^\ell)\) and \(F_1(\zeta; q) := \prod_{n=1}^{\infty} (1 - \zeta q^n)\).
Proof for crank

As $z \to 0$ in $D_\theta$, for $q = e^{-z}$ and $\zeta$ a primitive $b$-th root of unity (Bringmann–Craig–Males–Ono)

$$F_1 (\zeta; e^{-z}) = \frac{1}{\sqrt{1-\zeta}} e^{-\frac{\zeta \Phi(\zeta, 2, 1)}{z}} (1 + O(|z|)),$$
Proof for crank

As $z \to 0$ in $D_\theta$, for $q = e^{-z}$ and $\zeta$ a primitive $b$-th root of unity (Bringmann–Craig–Males–Ono)

$$F_1 (\zeta; e^{-z}) = \frac{1}{\sqrt{1 - \zeta}} e^{-\frac{\zeta \Phi(\zeta, 2, 1)}{z}} (1 + O(|z|)),$$

where $\Phi$ is the Lerch’s transcendent

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}.$$
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On the major arc (Bringmann–Dousse)

\[
(e^{-z}; e^{-z})^{-1}_\infty = \sqrt{\frac{z}{2\pi}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)),
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Proof for crank

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On the major arc (Bringmann–Dousse)

$$\left( e^{-z}; e^{-z} \right)_\infty^{-1} = \sqrt{\frac{z}{2\pi}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)),$$

while on the minor arc, for some $C > 0$

$$\left| \left( e^{-z}; e^{-z} \right)_\infty^{-1} \right| \leq x^{\frac{1}{2}} e^{\frac{\pi^2}{6x} - \frac{C}{x}}.$$
Proof for crank

Using the definition of $F_1(\zeta; q)$

\[
\left| \log \left( \frac{1}{F_1(\zeta; q)} \right) \right| = \left| \sum_{k \geq 1} \frac{\zeta^k}{k} \frac{q^k}{1 - q^k} \right|
\leq \left| \frac{\zeta q}{1 - q} \right| - \frac{|q|}{1 - |q|} + \log \left( \frac{1}{(|q|; |q|)_\infty} \right).
\]
Proof for crank

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$$\left| \log \left( \frac{1}{F_1(\zeta; q)} \right) \right| = \left| \sum_{k \geq 1} \frac{\zeta^k}{k} \frac{q^k}{1 - q^k} \right|$$

$$\leq \left| \frac{\zeta q}{1 - q} \right| - \left| \frac{q}{1 - |q|} \right| + \log \left( \frac{1}{(|q|; |q|)_\infty} \right).$$

$$\Rightarrow \quad \left| \frac{1}{F_1(\zeta; q)} \right| \ll e^{-c' x} (|q|; |q|)_\infty^{-1},$$

for some $c' > 0$. 
Proof for crank

Since an analogous calculation holds for $F_1(\zeta^{-1}; q)$ one may conclude that

$$\left| C\left(\zeta_b^j; q\right) \right| < \left| (q; q)_{\infty}^{-1} \right|$$

on the minor arc.
Proof for crank

Since an analogous calculation holds for $F_1(\zeta^{-1}; q)$ one may conclude that

$$\left| C \left( \zeta^j b; q \right) \right| < \left| (q; q)^{-1}_{\infty} \right|$$

on the minor arc.

For the major arc

$$C \left( \zeta; q \right) \ll e^{-\frac{\pi^2}{6} \text{Re}\left( \frac{1}{z} \right) + \text{Re}\left( \frac{\zeta \Phi(\zeta, 2, 1)}{z} \right) + \text{Re}\left( \frac{\zeta^{-1} \Phi(\zeta^{-1}, 2, 1)}{z} \right)}.$$
Therefore

\[ C \left( \zeta^j_b; q \right) = o \left( (q; q)^{-1} \right) \]

if and only if

\[
\left( \frac{\pi^2}{3} - \varepsilon - \phi_1 - \phi_1' \right) \frac{x}{|z|^2} > \left( \phi_2 + \phi_2' \right) \frac{y}{|z|^2},
\]
Therefore

\[ C \left( \zeta_j^b; q \right) = o \left( (q; q)^{-1}_\infty \right) \]

if and only if

\[
\left( \frac{\pi^2}{3} - \varepsilon - \phi_1 - \phi'_1 \right) \frac{x}{|z|^2} > (\phi_2 + \phi'_2) \frac{y}{|z|^2},
\]

where \( \phi_1 + i\phi_2 := \zeta_j^b \Phi(\zeta_j^b, 2, 1) \) and \( \phi'_1 + i\phi'_2 := \zeta_j^{-b} \Phi(\zeta_j^{-b}, 2, 1) \).
Proof for crank

Note that $\phi_1 = \frac{\pi^2}{6} - \frac{\pi^2 j}{b} \left(1 - \frac{j}{b}\right) = \phi'_1$ and $\phi_2 = -\phi'_2$. 
Proof for crank

Note that $\phi_1 = \frac{\pi^2}{6} - \frac{\pi^2 j}{b} \left(1 - \frac{j}{b}\right) = \phi'_1$ and $\phi_2 = -\phi'_2$.

Therefore, our assumption reduces to

$$\left(\frac{2\pi^2 j}{b} \left(1 - \frac{j}{b}\right) - \varepsilon\right) \frac{x}{|z|^2} > 0,$$

which holds, since we have $b > 0$, $1 \leq j \leq b - 1$ and $x = \text{Re}(z) > 0$. \qed
Proof for Betti numbers

Let $X$ be a Hilbert scheme

$$G_X(T; q) := \sum_{n \geq 0} P(X; T) q^n,$$

with $P(X; T) := \sum_j b_j(X) T^j$ the Poincaré polynomial.
Proof for Betti numbers

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Using orthogonality of roots of unity

$$\sum_{n \geq 0} B(a, b; X) q^n = \frac{1}{b} \sum_{r=0}^{b-1} \zeta_b^{-ar} G_X (\zeta_b^r; q).$$
Proof for Betti numbers

Boccalini’s thesis states that

\[
G_{X_1}(\zeta; q) = \sum_{n \geq 0} P(X_1; \zeta) q^n = \frac{1 + \zeta^2}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3(\zeta^2; q)^{-1},
\]
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where \( F_3(\zeta; q) := \prod_{n=1}^{\infty} (1 - \zeta^{-1}(\zeta q)^n). \)
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where \( F_3(\zeta; q) := \prod_{n=1}^{\infty} \left( 1 - \zeta^{-1}(\zeta q)^n \right) \).

We obtain

\[ H_{X_1}(a, b; q) := \sum_{n \geq 0} B(a, b; X_1) q^n \]

\[ = \frac{1}{b} \left( 1 + (-1)^a \delta_{2|b} \right) G_{X_1}(1; q) + \frac{1}{b} \sum_{0 < r \leq b-1 \atop r \neq \frac{b}{2}} \zeta_{b-ar}^{-1} G_{X_1}(\zeta_{rb}; q). \]
Proof for Betti numbers

Since

\[ G_{\chi_1}(1; e^{-z}) = \frac{2}{(1 - e^{-z})(1 - e^{-2z})} (e^{-z}; e^{-z})^{-1}_\infty \]

\[ = \left( \frac{1}{z^2} + \frac{3}{2z} + \frac{11}{12} + O(z) \right) (e^{-z}; e^{-z})^{-1}_\infty , \]
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the asymptotic behaviour is essentially controlled by the Pochhammer symbol.
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Using the asymptotic behaviour of \((q; q)_{\infty}\) we see that

\[ G_{X_1}(1; e^{-z}) = \frac{1}{\sqrt{2\pi z^2}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)). \]
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\[ G_{\chi_1}(1; e^{-z}) = \frac{1}{\sqrt{2\pi z}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)). \]

For \(\zeta_b^r \neq 1\) it is enough to show that on the major and minor arcs,

\[ G_{\chi_1}(\zeta_b^r; q) = o(G_{\chi_1}(1; q)). \]
Proof for Betti numbers

On the major arc (Bringmann–Craig–Males–Ono)

\[ F_3(\zeta^2r; e^{-z})^{-1} \ll e^{\frac{\pi^2}{6z}} |z|^{-N}, \]

for any \( N \in \mathbb{N} \)
On the major arc (Bringmann−Craig−Males−Ono)

\[ F_3(\zeta^{2r}; e^{-z})^{-1} \ll e^{\frac{\pi^2}{6z}|z|^{-N}}, \]

for any \( N \in \mathbb{N} \) and therefore we see that \( G_{X_1}(\zeta^r; q) = o(G_{X_1}(1; q)) \).
Proof for Betti numbers

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On the minor arc we obtain that

\[ \left| F_3 (\zeta_b^{2r}; q)^{-1} \right| < \left| (q; q)_{\infty}^{-1} \right| \]
On the major arc (Bringmann–Craig–Males–Ono)

\[ F_3\left(\zeta_{2r}^b; e^{-z}\right)^{-1} \ll e^{\frac{\pi^2}{6z}} |z|^{-N}, \]

for any \( N \in \mathbb{N} \) and therefore we see that \( G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q)) \).

On the minor arc we obtain that

\[ \left| F_3 \left( \zeta_b^{2r}; q \right)^{-1} \right| < \left| (q; q)_\infty^{-1} \right| \]

and therefore again \( G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q)) \).
Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi z}^3} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$
Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi} z^2} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$

We are left to apply BCMO with $A = \frac{\pi^2}{6}, B = -\frac{3}{2}$, and $\alpha_0 = \frac{d(a, b)}{\sqrt{2\pi}}$. 
Proof for Betti numbers

Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi z^2}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$

We are left to apply BCMO with $A = \frac{\pi^2}{6}, B = -\frac{3}{2},$ and $\alpha_0 = \frac{d(a, b)}{\sqrt{2\pi}}$ which yields that

$$B(a, b; X_1) = \frac{\sqrt{3}d(a, b)}{2\pi^2} e^{\pi \sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

from which one may also conclude asymptotic equidistribution.
Proof for Betti numbers

Similarly, it is known that

\[ G_{X_2}(\zeta; q) := \frac{1 + \zeta^2 - \zeta^2 q}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3 \left( \zeta^2; q \right)^{-1}, \]
Proof for Betti numbers

Similarly, it is known that

\[ G_{X_2}(\zeta; q) := \frac{1 + \zeta^2 - \zeta^2 q}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3 \left( \zeta^2; q \right)^{-1}, \]

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\]

\[
G_{X_3}(\zeta; q) := \frac{1}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3 (\zeta^2; q)^{-1},
\]

\[
G_{X_4}(\zeta; q) := F_3 (\zeta^2; q)^{-1} \prod_{j=1}^{m} \frac{1}{1 - \zeta^2 j q^j}.
\]
Proof for Betti numbers

Similarly, it is known that

\[
\begin{align*}
G_{X_2}(\zeta; q) &:= \frac{1 + \zeta^2 - \zeta^2 q}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3 (\zeta^2; q)^{-1}, \\
G_{X_3}(\zeta; q) &:= \frac{1}{(1 - \zeta^2 q)(1 - \zeta^4 q^2)} F_3 (\zeta^2; q)^{-1}, \\
G_{X_4}(\zeta; q) &:= F_3 (\zeta^2; q)^{-1} \prod_{j=1}^{m} \frac{1}{1 - \zeta^{2j} q^j}.
\end{align*}
\]

An analogous argument to the case of \( X_1 \) holds.
Thank you for your attention!