

Holomorphic Lagrangian Fibrations on Hypercomplex Manifolds

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A hypercomplex manifold is a manifold equipped with a triple of complex structures satisfying the quaternionic relations. A holomorphic Lagrangian variety on a hypercomplex manifold with trivial canonical bundle is a holomorphic subvariety which is calibrated by a form associated with the holomorphic volume form; this notion is a generalization of the usual holomorphic Lagrangian subvarieties known in hyperKähler geometry. An HKT (hyperKähler with torsion) metric on a hypercomplex manifold is a metric determined by a local potential, in a similar way to the Kähler metric. We prove that a base of a holomorphic Lagrangian fibration is always Kähler, if its total space is HKT. This is used to construct new examples of hypercomplex manifolds which do not admit an HKT structure.

1 Introduction

1.1 HKT metrics and $SL(n, \mathbb{H})$ -structures

Let I, J , and K be complex structures on a manifold M satisfying the quaternionic relation $I \circ J = -J \circ I = K$. Then (M, I, J, K) is called a *hypercomplex manifold*. Hypercomplex manifolds are quaternionic analogs of complex manifolds, in the same way as hyperKähler manifolds are quaternionic analogs of Kähler manifolds. However, the

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theory of hypercomplex manifolds is much richer in examples and (unlike that of hyperKähler manifolds) still not well understood.

For more details on hypercomplex and hyperKähler structures, please see Section 2.1.

The term *hypercomplex* is due to Boyer [4], who classified compact hypercomplex manifolds in $\dim_{\mathbb{H}} = 1$. These are: K3 surfaces, compact 2D complex tori, and Hopf surfaces. However, in dimension > 1 there is no classification apparent.

The notion of hypercomplex structure was considered as early as in 1955 by Obata [18], who proved existence and uniqueness of a torsion-free connection preserving a hypercomplex structure. However, the first nontrivial examples were obtained only in 1988 by physicists Spindel *et al.* [22], and the intensive study of hypercomplex structures began with the paper by Joyce [15], who classified hypercomplex structures on homogeneous spaces, and constructed one on each compact Lie group multiplied by a compact torus of appropriate dimension.

The main geometric tool allowing one to work with the hypercomplex manifolds is a notion of an HKT metric, due to physicists Howe and Papadopoulos [14]. For a definition and a discussion of HKT structure see Section 2.1. HKT structures were much used in physics since mid-1990s [7, 10, 11], and then mathematicians also started using HKT structures to study the hypercomplex manifolds.

In [25], the second-named author developed Hodge theory for HKT manifolds, and it turned out to be quite useful. To illustrate the usability of HKT metrics, let us quote the following result, obtained in [26]. Let (M, I, J, K) be a hypercomplex manifold with (M, I) a complex manifold of Kähler type. Then (M, I) is in fact hyperKähler (i.e., admits a hyperKähler structure). The idea of the proof is that any Kähler metric on (M, I) gives an HKT metric on (M, I, J, K) , and this metric can be used to apply Hodge theory. The interplay between Kähler geometry and HKT geometry is quite extensive, and can be used to study the geometry of hypercomplex manifolds extensively.

In the present paper, we obtain a result in a similar vein, constructing a Kähler metric on a base of holomorphic Lagrangian fibration on an HKT manifold.

Of course, the notion of “holomorphic Lagrangian fibration” is itself highly non-trivial, because an HKT manifold is not necessarily holomorphically symplectic. It was developed in [9], using the theory of calibrations and the holonomy of the Obata connection.

The Obata connection is an immensely powerful tool, known since 1950s [18], but even the most primitive invariants of Obata connection, such as its holonomy, are hard to compute, and still unknown in most cases. When (M, I, J, K) admits a hyperKähler

metric, Obata connection is equal to the Levi-Civita connection of the hyperKähler metric and its holonomy is $\mathrm{Sp}(n)$. The converse is also true: whenever holonomy of a torsion-free connection is $\mathrm{Sp}(n)$, it is a Levi-Civita connection of a hyperKähler manifold.

In a general situation, it is hard to say anything definite about the holonomy $\mathrm{Hol}(\nabla)$ of the Obata connection. It is clear that $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(\mathbb{H}, n)$, but the equality is rarely realized. In fact, before the publication of the paper [20] in 2011, there was no single example of a hypercomplex manifold with $\mathrm{Hol}(\nabla) = \mathrm{GL}(\mathbb{H}, n)$. In [20], the first-named author provided an example of such a manifold, by proving that $\mathrm{Hol}(\nabla) = \mathrm{GL}(\mathbb{H}, 2)$ for the homogeneous hypercomplex structure on $\mathrm{SU}(3)$, constructed by Joyce [15].

However, there are many examples of hypercomplex manifolds with $\mathrm{Hol}(\nabla) \subsetneq \mathrm{GL}(\mathbb{H}, n)$. Two biggest known families of hypercomplex manifolds are the Joyce's homogeneous manifolds from [15] and hypercomplex nilmanifolds [1]. As shown in [2], for all nilmanifolds, the holonomy of the Obata connection lies in

$$\mathrm{SL}(\mathbb{H}, n) = [\mathrm{GL}(\mathbb{H}, n), \mathrm{GL}(\mathbb{H}, n)] \subset \mathrm{GL}(\mathbb{H}, n).$$

A hypercomplex manifold with holonomy in $\mathrm{SL}(n, \mathbb{H})$ is called an *$\mathrm{SL}(n, \mathbb{H})$ -manifold*. Such manifold might have holonomy group which is strictly less than $\mathrm{SL}(n, \mathbb{H})$; in fact, no example of a manifold with $\mathrm{Hol}(\nabla) = \mathrm{SL}(\mathbb{H}, n)$ is known so far.

For any $\mathrm{SL}(n, \mathbb{H})$ -manifold, the Obata connection on the canonical bundle $K(M, I) = \Lambda^{2n,0}(M, I)$ of (M, I) preserves a nontrivial section. This implies that $K(M, I)$ is holomorphically trivial (see [28]). In the presence of an HKT metric, the converse is also true: any compact hypercomplex manifold with holomorphically trivial canonical bundle $K(M, I)$ and an HKT metric satisfies $\mathrm{Hol}(\nabla) \subset \mathrm{SL}(\mathbb{H}, n)$. This result is obtained in [28] using the Hodge theory for HKT manifolds developed in [25].

The Hodge-theoretic constructions of [25] work for $\mathrm{SL}(n, \mathbb{H})$ -manifolds with HKT-structure especially well. For such manifolds, one obtains Hodge-type decomposition on the holomorphic cohomology bundle $H^*(\mathcal{O}_{(M,I)})$.

1.2 Calibrations on manifolds and Lagrangian fibrations

Let M be a Riemannian manifold. A *calibration* on M is a closed k -form η , such that $\eta(x_1, \dots, x_k) \leq 1$ for each orthonormal k -tuple $x_1, \dots, x_k \in TM$. A k -dimensional oriented subspace $V \subset T_x M$ is called *calibrated* if the Riemannian volume of V is equal to $\eta|_V$, and a subvariety $Z \subset M$ of dimension k is called *calibrated* if its singularities are of Hausdorff codimension ≥ 1 , and all smooth tangent planes $T_z Z \subset T_z M$ are calibrated.

The theory of calibrations has a long and distinguished history, starting from the work of Harvey and Lawson [12]. In [9], several families of calibrations were constructed on hyperKähler manifolds, using the quaternionic linear algebra. Let (M, I, J, K) be a hyperKähler manifold, and $\omega_I, \omega_J,$ and ω_K its Kähler forms. These forms generate an interesting commutative subalgebra in $\Lambda^*(M)$ (see [9, 24] for structure results about this algebra). Grantcharov and Verbitsky [9] discovered several new calibrations which are expressed as polynomials of $\omega_I, \omega_J,$ and ω_K . One of these calibrations, denoted as Ψ in the sequel (see (2.2)), is called *holomorphic Lagrangian calibration*. On a hyperKähler manifold, it calibrates holomorphic Lagrangian subvarieties in (M, I) .

It is surprising (and quite wonderful) that this form remains closed, even when the metric is not hyperKähler, provided that the holonomy of the Obata connection of (M, I, J, K) belongs to $\mathrm{SL}(n, \mathbb{H})$. The hyperKähler condition can be weakened drastically: it suffices to assume that the metric g is quaternionic Hermitian, and the Obata-parallel section of $K(M, I)$ has constant length with respect to g . In this case Ψ is closed. If rescaled properly, this form is a calibration, calibrating a new class of complex subvarieties of (M, I) called *holomorphic Lagrangian* (see [9] and Definition 3.1 for more details).

In [21], it was shown that holomorphic Lagrangian subvarieties of $\mathrm{SL}(n, \mathbb{H})$ -manifolds exist only in a countable number of complex structure of the form $L = aI + bJ + cK$ (such complex structures are called *induced by a hypercomplex structure*, see Section 2.1). In the present paper, we study holomorphic Lagrangian fibrations on hypercomplex manifolds. Such fibrations are often present in examples ([9]; see also Section 3.2).

The holomorphic Lagrangian fibrations are of significant interest for mathematicians and physicists; for an early survey of the subject, please see the paper [19], and for applications, see e.g. [16]. The non-Kähler version of this geometry defined above should be even more useful, because the number of examples is much greater.

In the present paper, we consider HKT metrics on $\mathrm{SL}(n, \mathbb{H})$ -manifolds admitting holomorphic Lagrangian fibrations. We prove that a base of such fibration is always Kähler, if the fibration is smooth (3.4). This allows us to construct new examples of hypercomplex manifolds admitting no HKT metrics.

Originally, it was conjectured that any compact hypercomplex manifold admits an HKT metric. Examples of hypercomplex nilmanifolds not admitting HKT metrics were constructed by Fino and Grantcharov [6]. Since then, hypercomplex nilmanifolds admitting HKT metrics were classified completely in [2]. It was found that in fact most of hypercomplex nilmanifolds are not HKT. However, the hypercomplex manifolds not

admitting an HKT metrics are still rare. Another example of a non-HKT manifold is given by Swann in [23, Theorem 5.8], using twisted torus bundle over a K3 surface.

2 Preliminaries

2.1 Hypercomplex manifolds and quaternionic Dolbeault complex

A C^∞ -manifold M is called *hypercomplex* if M is equipped with complex structures $I, J,$ and $K,$ that satisfy the quaternionic relations

$$IJ = -JI = K.$$

It was shown in [18] that any hypercomplex manifold admits a unique torsion-free connection that preserves $I, J,$ and $K.$ This connection is called *the Obata connection.* Note that the holonomy of the Obata connection is a subgroup of $GL(n, \mathbb{H}).$

Any almost-complex structure which is preserved by a torsion-free connection is integrable (this follows from Newlander–Nirenberg theorem). Therefore, for any $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 1,$ the almost-complex structure $L = aI + bJ + cK$ is integrable. We denote by (M, L) the corresponding complex manifold. Such complex structures are called *induced by quaternions.*

Let (M, I, J, K) be a hypercomplex manifold of real dimension $4n.$ The hypercomplex structure induces the action of $SU(2)$ on all tensor bundles over $M.$ Recall that any irreducible complex representation of $SU(2)$ is of the form $S^k U,$ where U is the standard 2D representation and S^k denotes the symmetric power. We will refer to any representation of the form $(S^k(U))^{\oplus m}$ (for arbitrary m) as a *weight k* representation of $SU(2).$

We make the following observation about the weight decomposition of the exterior algebra $\Lambda_{\mathbb{C}}^* M = \Lambda^* M \otimes_{\mathbb{R}} \mathbb{C}.$ First, note that $\Lambda_{\mathbb{C}}^1 M$ is an $SU(2)$ representation of weight one. It follows from the Clebsch–Gordan formula that $\Lambda_{\mathbb{C}}^k M$ is a sum of representations of weight $\leq k.$ By duality, the same is true about $\Lambda_{\mathbb{C}}^{4n-k} M.$ Denote by $\Lambda_+^k M$ the maximal subrepresentation of weight k for $k \leq 2n$ and of weight $4n - k$ for $k > 2n$ inside $\Lambda_{\mathbb{C}}^k M.$ Denote by $\Pi_+ : \Lambda^* M \rightarrow \Lambda_+^* M$ the equivariant projection onto the component of maximal weight.

Note that the Hodge decomposition with respect to an arbitrary complex structure (we can pick I without loss of generality) is compatible with the weight decomposition:

$$\Lambda_+^k M = \bigoplus_{p+q=k} \Lambda_{I,+}^{p,q} M,$$

where $\Lambda_{I,+}^{p,q}M = \Lambda_I^{p,q}M \cap \Lambda_+^k M$. This is actually a weight decomposition of an $SU(2)$ -representation for a special choice of Cartan subalgebra corresponding to I . Therefore, $\Lambda_+^k M$ is generated by $\Lambda_{I,+}^{k,0}M = \Lambda_I^{k,0}M$ as an $SU(2)$ -module. It follows that the summands $\Lambda_{I,+}^{p,q}M$ are of the same dimension and actually isomorphic to $\Lambda_I^{p+q,0}M$. We will denote these isomorphisms by

$$\mathcal{R}_{p,q}: \Lambda_I^{p+q,0}M \xrightarrow{\sim} \Lambda_{I,+}^{p,q}M,$$

see [27] and formula (2.3) below.

Given a hyperhermitian metric g on M , we can consider the corresponding Kähler forms ω_I, ω_J , and ω_K . It is easy to check that the form

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K$$

lies in $\Lambda_I^{2,0}M$. If $d\Omega_I = 0$, then the manifold M is called *hyperKähler*. This condition implies that g is Kähler with respect to any induced complex structure. If Ω_I satisfies a strictly weaker condition $\partial\Omega_I = 0$, then M is called an *HKT manifold*, and g is called an *HKT metric* (HKT stands for HKT; see [8] for a general introduction to HKT geometry).

We can reformulate the HKT condition as follows. Denote by $d_+ : \Lambda_+^k M \rightarrow \Lambda_+^{k+1} M$ the composition $\Pi_+ \circ d$ of the de Rham differential and the projection onto the component of maximal weight. Note that $\omega_I \in \Lambda_{I,+}^{1,1}M$. Then by [25, Theorem 5.7], the condition $\partial\Omega_I = 0$ is equivalent to

$$d_+\omega_I = 0. \tag{2.1}$$

That is, for a HKT metric g the exterior differential of its Kähler form $d\omega_I$ has weight one. This observation will be important for the proof of the main theorem.

2.2 $SL(n, \mathbb{H})$ -manifolds and holomorphic Lagrangian calibration

As was mentioned above, the holonomy of the Obata connection on a hypercomplex manifold is a subgroup of $GL(n, \mathbb{H})$. Recall that this group can be defined as follows. Consider a vector space V of real dimension $4n$ with a quaternionic structure I, J , and K . Then $GL(n, \mathbb{H})$ consists of those linear transformations of V that commute with I, J , and K . Consider the Hodge decomposition $V_{\mathbb{C}} = V \otimes \mathbb{C} = V_I^{1,0} \oplus V_I^{0,1}$. The action of $GL(n, \mathbb{H})$ preserves this decomposition, hence it induces an action on all exterior powers of $V_I^{1,0}$. Denote by $SL(n, \mathbb{H})$ the subgroup of $GL(n, \mathbb{H})$ consisting of those elements that act identically on $V_I^{2n,0}$.

Consider a hypercomplex manifold (M, I, J, K) . If the holonomy of the Obata connection is contained in $\mathrm{SL}(n, \mathbb{H})$, then we call M an $\mathrm{SL}(n, \mathbb{H})$ -manifold. It follows almost immediately from the definition (see [28, Claim 1.1]), that on an $\mathrm{SL}(n, \mathbb{H})$ -manifold the canonical bundle $\Lambda_I^{2n,0} M$ is flat with respect to the Obata connection. It also follows [28, Claim 1.2], that any parallel section of $\Lambda_I^{2n,0} M$ is holomorphic. Moreover, if the manifold (M, I, J, K) is HKT and compact, the condition $\mathrm{Hol}(M) \subset \mathrm{SL}(n, \mathbb{H})$ is equivalent to holomorphic triviality of the canonical bundle [28, Theorem 2.3].

Denote by $\Phi_I \in \Lambda_I^{2n,0} M$ a parallel section which we call a *holomorphic volume form*. Note that the operator J defines a real structure (that is, a complex-antilinear involution) on $\Lambda_I^{2n,0} M$ by $\eta \mapsto \overline{J\eta}$. We can always assume that Φ_I is real with respect to this structure.

Given an $\mathrm{SL}(n, \mathbb{H})$ -manifold M with a hyperhermitian metric g , denote by Φ_I the holomorphic volume form and by $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ the $(2, 0)$ -form associated with the metric. In [9], a number of calibrations (in the sense of [12]) on $\mathrm{SL}(n, \mathbb{H})$ -manifolds were constructed. In particular, it was shown that the form

$$\Psi = (-\sqrt{-1})^n \mathcal{R}_{n,n}(\Phi_I) \in \Lambda_{I,+}^{n,n} M \quad (2.2)$$

is a calibration with respect to a properly rescaled metric. It was shown that this form calibrates Ω_I -Lagrangian subvarieties of M . Recall, that a complex (with respect to the complex structure I) subvariety $N \subset M$ of complex dimension n is called *Ω_I -Lagrangian* if the restriction of Ω_I to the smooth part of N vanishes. This condition is equivalent to the following: $T_x N$ is orthogonal to $J(T_x N)$ with respect to the metric g at any smooth point $x \in N$. To see this, observe that for any vector fields $X, Y \in T_I^{1,0} M$, we have $\Omega_I(X, Y) = 2g(JX, Y)$.

We will not need the construction from [9] in its full generality, but only some basic properties of the form Ψ . For the convenience of the reader, we will formulate these properties in the following lemma and give a short proof, independent of [9].

Lemma 2.1. Let (M, I, J, K) be an $\mathrm{SL}(n, \mathbb{H})$ -manifold of real dimension $4n$, $\Phi_I \in \Lambda_I^{2n,0} M$ a holomorphic volume form and $\Psi = (-\sqrt{-1})^n \mathcal{R}_{n,n}(\Phi_I) \in \Lambda_{I,+}^{n,n} M$. Let $N \subset M$ be an I -complex subvariety, $\dim_{\mathbb{C}} N = n$, with $T_x N \cap J(T_x N) = 0$ at all smooth points $x \in N$. Then:

- (1) $d\Psi = 0$,
- (2) $\Psi|_N$ is a strictly positive volume form on the smooth part of N . □

Proof. (1) As was mentioned above (see also formula (2.3)), the operator $\mathcal{R}_{n,n}$ is induced by the $SU(2)$ -action on the exterior algebra of M . Since the Obata connection ∇ preserves the hypercomplex structure, it commutes with the $SU(2)$ -action and with the operator $\mathcal{R}_{n,n}$. It was explained above that $\nabla\Phi_I = 0$, thus $\nabla\Psi = 0$ also. The Obata connection is torsion-free, hence $\nabla\Psi = 0$ implies $d\Psi = 0$.

(2) Let $x \in N$ be a smooth point. By the assumptions of the lemma, we can choose a basis e_1, \dots, e_n of $T_{I,x}^{1,0}N$, such that $e_1, \dots, e_n, J\bar{e}_1, \dots, J\bar{e}_n$ will form a basis of $T_{I,x}^{1,0}M$. To show that $\Psi|_N$ is strictly positive, we have to evaluate the form Ψ on the polyvector $\xi = (-\sqrt{-1})^n e_1 \wedge \bar{e}_1 \wedge \dots \wedge e_n \wedge \bar{e}_n \in \Lambda_I^{n,n}(T_x M)$.

We will use the following explicit description of the operator $\mathcal{R}_{n,n}$. Consider the operators: $\mathcal{H} = -\sqrt{-1}I$, $\mathcal{X} = \frac{1}{2}(\sqrt{-1}K - J)$ and $\mathcal{Y} = \frac{1}{2}(\sqrt{-1}K + J)$ acting on $\Lambda_{\mathbb{C}}^1 M$. It is straightforward to check that these operators satisfy the standard $\mathfrak{sl}_2(\mathbb{C})$ commutator relations: $[\mathcal{X}, \mathcal{Y}] = \mathcal{H}$, $[\mathcal{H}, \mathcal{X}] = 2\mathcal{X}$, $[\mathcal{H}, \mathcal{Y}] = -2\mathcal{Y}$. Also observe that

$$\begin{aligned} \mathcal{H}|_{\Lambda_I^{1,0}M} &= 1, & \mathcal{X}|_{\Lambda_I^{1,0}M} &= 0, & \mathcal{Y}|_{\Lambda_I^{1,0}M} &= J, \\ \mathcal{H}|_{\Lambda_I^{0,1}M} &= -1, & \mathcal{X}|_{\Lambda_I^{0,1}M} &= -J, & \mathcal{Y}|_{\Lambda_I^{0,1}M} &= 0. \end{aligned}$$

We can extend \mathcal{H} , \mathcal{X} , and \mathcal{Y} as derivations to the whole exterior algebra. Then we have

$$\mathcal{R}_{n,n} = \mathcal{Y}^n \tag{2.3}$$

and

$$\langle \Psi, \xi \rangle = (-\sqrt{-1})^n \langle \mathcal{Y}^n \Phi_I, \xi \rangle = (\sqrt{-1})^n \langle \Phi_I, \mathcal{Y}^n \xi \rangle.$$

Observe, that $\mathcal{Y}|_{T_I^{1,0}M} = 0$, $\mathcal{Y}|_{T_I^{0,1}M} = J$, and since \mathcal{Y} acts as a derivation, we have

$$\mathcal{Y}^n \xi = n! (-\sqrt{-1})^n e_1 \wedge J\bar{e}_1 \wedge \dots \wedge e_n \wedge J\bar{e}_n.$$

The volume form is given by $\Phi_I = a e_1^* \wedge J\bar{e}_1^* \wedge \dots \wedge e_n^* \wedge J\bar{e}_n^*$ for some positive real number a , and we see that $\langle \Psi, \xi \rangle > 0$. ■

3 Holomorphic Lagrangian Fibrations on $SL(n, \mathbb{H})$ -Manifolds

3.1 HKT structures and Lagrangian fibrations

Definition 3.1. A *holomorphic Lagrangian subvariety* of an $SL(n, \mathbb{H})$ -manifold is a subvariety calibrated by the form Ψ of Lemma 2.1. □

Definition 3.2. Let M be an $\mathrm{SL}(n, \mathbb{H})$ -manifold, and $\phi: (M, I) \rightarrow X$ a smooth holomorphic fibration. It is called a *holomorphic Lagrangian fibration* if all its fibers are holomorphic Lagrangian subvarieties. We will say that the fibration $\phi: (M, I) \rightarrow X$ is *smooth* if ϕ is a submersion (hence all the fibers are smooth). \square

Remark 3.3. Suppose that we are given an arbitrary holomorphic fibration $\phi: (M, I) \rightarrow X$ with M compact and $\dim_{\mathbb{C}} X = \frac{1}{2} \dim_{\mathbb{C}} M$. Denote by F_x the fiber of ϕ over $x \in X$. Consider the set $U = \{x \in X: J(TF_x) \cap TF_x = 0\}$. It is clear that U is open and all the fibers over U can be made Lagrangian with respect to some hyperhermitian metric. Namely, the condition $J(TF_x) \cap TF_x = 0$ implies that each fiber of the vector bundle $J(TF_x)$ projects onto $T_x X$ by ϕ , so that we can lift any Hermitian metric from U to $J(TF_x)$. Then we can uniquely extend it to a hyperhermitian metric on $\phi^{-1}(U)$ with $J(TF_x)$ orthogonal to TF_x . This means that the set of Lagrangian fibers of ϕ is open in M . \square

The main result of this paper is the following theorem.

Theorem 3.4. Let M be a compact $\mathrm{SL}(n, \mathbb{H})$ -manifold, and $\phi: (M, I) \rightarrow X$ a smooth holomorphic Lagrangian fibration. Assume that M admits an HKT-structure. Then X is Kähler. \square

Proof. Let $\omega_I \in \Lambda_I^{1,1} M$ be a Kähler form, and Ψ the holomorphic Lagrangian calibration defined by (2.2). Consider the form

$$\Theta = \Psi \wedge \omega_I \in \Lambda_I^{n+1, n+1} M.$$

By Lemma 2.1, Ψ is closed, so we have $d\Theta = \Psi \wedge d\omega_I \in \Lambda^{2n+3} M$. It follows from (2.1) that $d\omega_I$ has weight one, hence Clebsch–Gordan formula implies that $d\Theta$ is a sum of weight $2n+1$ and weight $2n-1$ components. But the weight decomposition of $\Lambda^{2n+3} M$ has components only up to weight $2n-3$, so $d\Theta$ has to vanish.

We will obtain a Kähler metric on X as a push-forward $\pi_* \Theta$, which is a $(1, 1)$ -current on X . We have to show that this current is strictly positive and smooth.

Consider any Hermitian metric on X and denote by η the corresponding $(1, 1)$ -form. It is always possible to rescale η so that it satisfies the condition $\pi^* \eta \leq \omega_I$. In this case, we have $\Theta = \Psi \wedge \omega_I \geq \Psi \wedge \pi^* \eta$. The inequality is preserved under taking push-forwards, so we have $\pi_* \Theta \geq \pi_*(\Psi \wedge \pi^* \eta)$.

For any test-form $\alpha \in \Lambda_I^{n-1, n-1} X$, we have by definition

$$\begin{aligned} \int_X \pi_*(\Psi \wedge \pi^*\eta) \wedge \alpha &= \int_M \Psi \wedge \pi^*\eta \wedge \pi^*\alpha \\ &= \int_M \Psi \wedge \pi^*(\eta \wedge \alpha) = \int_X \pi_*\Psi \wedge \eta \wedge \alpha. \end{aligned}$$

Thus, we have the inequality $\pi_*\Theta \geq \pi_*(\Psi) \wedge \eta$ where $\pi_*(\Psi)$ is a closed 0-current on X , that is a constant. To see that this constant is positive, take $\alpha = \eta^{n-1}$ and observe that $\Psi \wedge \pi^*(\eta^n)$ is a smooth positive form of top degree on M . It suffices to check that this form is nonzero at some point of M . Pick any noncritical point x of π and decompose the tangent space $T_x M = V_0 \oplus V_1$, where V_0 is tangent to the fiber and V_1 is any complementary subspace. Since $\pi^*\eta^n|_{V_0} = 0$ and $\pi^*\eta^n|_{V_1}$ is a volume form on V_1 , we should check that $\Psi|_{V_0}$ is nonzero. But this follows from the second statement in Lemma 2.1, since the fiber is Lagrangian.

It remains to check that $\pi_*\Theta$ is smooth. This follows from smoothness of the fibration $\phi: (M, I) \rightarrow X$. Since ϕ is a submersion, we can choose a splitting $TM = \mathcal{V} \oplus \mathcal{H}$, where \mathcal{V} is a subbundle, tangent to the fibers of ϕ . Using this splitting, we can lift vector field from X to M . Denote by F_x the fiber $\phi^{-1}(x)$ over a point $x \in X$. Consider a $(1, 1)$ -form θ on X , which is given by

$$\theta(\xi, \bar{\eta})_x = \int_{F_x} (\pi^*\xi) \lrcorner (\pi^*\bar{\eta}) \lrcorner \Theta,$$

where $\xi, \eta \in T^{1,0}X$ and $\pi^*\xi, \pi^*\eta$ denote the lifts of these vector fields to \mathcal{H} . Note, that θ does not actually depend on the choice of \mathcal{H} , since convolution of Θ with a vertical vector field from \mathcal{V} gives a differential form that restricts trivially to the fiber. The form θ coincides with $\pi_*\Theta$ as a current, thanks to Fubini theorem. But it is clear from the definition that θ is smooth. This completes the proof. \blacksquare

Remark 3.5. Note that for a nonsmooth holomorphic Lagrangian fibration $\phi: (M, I) \rightarrow X$ the above proof shows that $\pi_*(\Theta \wedge \omega_I)$ is a Kähler current. Therefore, X is in Fujiki class \mathcal{C} , whenever M admits an HKT-structure (see [5]). \square

3.2 Examples of Lagrangian fibrations

In this subsection, we construct a class of hypercomplex $SL(n, \mathbb{H})$ -manifolds that do not admit an HKT metric. The idea, which was also used in [17], is to consider a torus fibration over an affine base.

Let (X, I) be a complex manifold, $\dim_{\mathbb{C}} X = n$. We call X an *affine complex manifold* if there exists a torsion-free connection $D: TX \rightarrow TX \otimes \Lambda^1 X$ which is flat and preserves the complex structure: $DI = 0$.

Fix a point $x \in X$ and consider the holonomy group $\text{Hol}_x(D) \subset \text{GL}(T_x X)$. We call X an affine manifold *with integer monodromy* if there exists a lattice $\Lambda_x \subset T_x X$ which is preserved by holonomy: $\text{Hol}_x(D)\Lambda_x = \Lambda_x$. In this case, we can construct a subset $\Lambda \subset TX$ in the total space of the tangent bundle, obtained as parallel translation of Λ_x . For any point $y \in X$ the intersection $\Lambda \cap T_y X$ is a lattice in $T_y X$.

Let $M = TX/\Lambda$, that is, a manifold obtained as a fiberwise quotient of TX . We will introduce a pair of anticommuting complex structures on TX that will descend to M . The connection D defines a splitting

$$T(TX) = \mathcal{V} \oplus \mathcal{H} \tag{3.1}$$

into a direct sum of the vertical subbundle \mathcal{V} which is tangent to the fibers of the projection $TX \rightarrow X$, and a horizontal complement \mathcal{H} . Note that for any point $(x, v) \in TX$ with $v \in T_x X$ we have natural isomorphisms $\mathcal{H}_{(x,v)} \simeq T_x X \simeq \mathcal{V}_{(x,v)}$, thus we can identify the components of the splitting (3.1). This also shows that the complex structure I acts naturally on both \mathcal{H} and \mathcal{V} . With these observations in mind, define a pair of operators from $\text{End}(T(TM))$:

$$\mathcal{I} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix},$$

where the block-matrix form corresponds to the decomposition (3.1). It is clear that these operators define a pair of anticommuting almost-complex structures. We need to check that these structures are integrable. This can be done locally.

Using the fact that the connection D is flat and I is parallel, we can choose a parallel local frame in TX of the form $e_1, Ie_1, \dots, e_n, Ie_n$. Since D is torsion-free, these vector fields commute, hence they define a local coordinate system on X . Since the sections e_i and Ie_i are flat, they are tangent to the subbundle \mathcal{H} . Therefore, they also define a local coordinate system in the total space of TX in which the complex structures \mathcal{I} and \mathcal{J} act as standard complex structures of \mathbb{H}^n . Thus, they are integrable.

This construction gives a hypercomplex structure on the total space of TX . Observe that this structure is invariant with respect to translations along the fiber. This implies that the hypercomplex structure descends to M . We call the manifold M constructed this way *quaternionic double* of an affine complex manifold X .

The hypercomplex structure on the quaternionic double M is locally modeled on \mathbb{H}^n by construction. Thus, the Obata connection on this manifold is flat (and it is actually induced from the flat connection D on X). Since the connection D preserves the lattice in the tangent bundle, it also preserves the holomorphic volume form. To see this, we can again choose a local frame $e_1, Ie_1, \dots, e_n, Ie_n$ as above. We can assume that the elements of this frame form a local basis for the lattice Λ preserved by D . Then the volume form $(e_1^* - \sqrt{-1}Ie_1^*) \wedge \dots \wedge (e_n^* - \sqrt{-1}Ie_n^*)$ is well-defined globally and parallel with respect to D , hence holomorphic. It follows that the Obata connection on M also preserves an I -holomorphic volume form. We conclude that the manifold M is an $\mathrm{SL}(n, \mathbb{H})$ -manifold.

Theorem 3.6. Let M be a quaternionic double of an affine complex manifold X . If X is not Kähler, then M does not admit an HKT metric. \square

Proof. We can choose a Hermitian metric g on X . Then we can define a hyperhermitian metric $h = g \oplus g$ on M , using the decomposition (3.1). With respect to this metric the fibers of the projection $M \rightarrow X$ are Lagrangian, hence we can apply Theorem 3.4. \blacksquare

We conclude this section by some examples of affine complex manifolds with integer monodromy. We remark that the manifolds in the examples will be non-Kähler (actually any complex affine manifold admitting a Kähler metric is a quotient of a torus, as follows from the Calabi–Yau theorem and the Bieberbach’s solution of Hilbert’s 18th problem; see e.g., [3]). Therefore, the corresponding quaternionic doubles possess no HKT metrics.

Example 3.7. Let N be a 3D real Heisenberg group, that is the group of upper-triangular unipotent 3×3 matrices with real elements. Consider the nilpotent Lie group $G = N \times \mathbb{R}$. Then G is diffeomorphic to \mathbb{C}^2 and one can check (see [13, Example 2]) that the structure of the Lie group on \mathbb{C}^2 can be explicitly given by

$$(w_1, w_2) \cdot (z_1, z_2) = (w_1 + z_1, w_2 - \sqrt{-1}\bar{w}_1 z_1 + z_2).$$

It is clear from this formula, that left translations are holomorphic affine transformations of \mathbb{C}^2 , so G has a left-invariant complex structure which is preserved by a flat torsion-free connection. We can choose a lattice in \mathbb{C}^2 , say $\Lambda = \mathbb{Z}[\sqrt{-1}]^2$. Then the manifold $X = \Lambda \backslash G$ (which is called a *primary Kodaira surface*) will be an affine complex manifold with integer monodromy. \square

Example 3.8. Let G be a 3D complex Heisenberg group. It can be described (see [13]) as \mathbb{C}^3 with multiplication given by

$$(w_1, w_2, w_3) \cdot (z_1, z_2, z_3) = (w_1 + z_1, w_2 + z_2, w_3 + z_3 + w_1 z_2).$$

We can choose a lattice in G , for example, $\Lambda = \mathbb{Z}[\sqrt{-1}]^3$ and take a quotient $X = \Lambda \backslash G$. This is an affine complex manifold with integer monodromy (which is called *an Iwasawa manifold*). \square

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